

An entanglement measure for n qubits¹

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Abstract

In Phys. Rev. A 61, 052306 (2000), Coffman, Kundu and Wootters introduced the residual entanglement for three qubits. In this paper, we present the entanglement measure $\tau(\psi)$ for even n qubits; for odd n qubits, we propose the residual entanglement $\tau^{(i)}(\psi)$ with respect to qubit i and the odd n -tangle $R(\psi)$ by averaging the residual entanglement with respect to each qubit. In this paper, we show that these measures are LU -invariant, entanglement monotones, invariant under permutations of the qubits, and multiplicative in some cases.

Keywords: entanglement measure, entanglement monotone, residual entanglement

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1 Introduction

Entanglement plays an important role in quantum computation and quantum information [1][2]. Many researchers in quantum information theory show interests in entanglement measures. Wootters introduced the idea of concurrence for two qubits to quantify entanglement [3]. Subsequently, the concurrence was further developed in [4][5][6]. Recently, Coffman, Kundu, and Wootters presented the residual entanglement which measures the amount of entanglement between subsystem A and subsystems BC for a tripartite state and

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gave an elegant expression for computing the residual entanglement for three qubits via the concurrence [7]. Vidal proposed entanglement monotone in [8]. It was later proved that the residual entanglement for three qubits is an entanglement monotone [9]. Recently, many authors have studied the residual entanglement.

Wong and Christensen defined even n -tangle for even n qubits which is invariant under permutations of the qubits and demonstrated that the even n -tangle for even n qubits is an entanglement monotone [10]. Their even n -tangle for even n qubits is listed as follows. See (2) in [10].

$$\tau_{1\dots n} = 2 \left| \sum a_{\alpha_1\dots\alpha_n} a_{\beta_1\dots\beta_n} a_{\gamma_1\dots\gamma_n} a_{\delta_1\dots\delta_n} \times \epsilon_{\alpha_1\beta_1} \epsilon_{\alpha_2\beta_2} \dots \epsilon_{\alpha_{n-1}\beta_{n-1}} \epsilon_{\gamma_1\delta_1} \epsilon_{\gamma_2\delta_2} \dots \times \epsilon_{\gamma_{n-1}\delta_{n-1}} \epsilon_{\alpha_n\gamma_n} \epsilon_{\beta_n\delta_n} \right|.$$

The even n -tangle is quartic and requires $3 * 2^{4n}$ multiplications. Our entanglement measure $\tau(\psi)$ for even n qubits is quadratic and requires 2^{n-1} multiplications [11]. Furthermore, Wong and Christensen indicated that the even n -tangle for even n qubits is not a measure of n -way entanglement [10]. The n -way entanglement is the entanglement that critically involves all n particles [10]. For odd n qubits, they said that the n -tangle is undefined for odd $n > 3$, see their abstract in [10].

In a separate work [12], Yu and Song defined the residual entanglement for n qubits as follows.

$$\tau_{ABC\dots N} = \min\{\tau_\alpha | \alpha = 1, 2, \dots, \sum_{i=1}^{[N/2]} C_N^i\}, \quad (1.1)$$

where α corresponds to all the possible foci and $C_N^i = n! / [(n-i)!i!]$. However, they did not show whether the residual entanglement is LU -invariant, or invariant under permutations of the qubits. Nor did they show that the residual entanglement is an entanglement monotone.

In another paper, Osterloh and Siewert constructed an n -qubit entanglement monotone from antilinear operators [13].

In an interesting work [14], Ou and Fan found that the monogamy of concurrence implies the monogamy of negativity, and that the resulting residual entanglement obtained through symmetrization all possible subsystem permutation gives rise to an entanglement monotone. In [14], they defined the negativity $\mathcal{N} = (||\rho^{T_A}|| - 1)/2$, where ρ^{T_A} is the partial transpose with respect to the subsystem A . Then, they defined the residual entanglements $\pi_A = \mathcal{N}_{A(BC)}^2 - \mathcal{N}_{AB}^2 - \mathcal{N}_{AC}^2$, $\pi_B = \mathcal{N}_{B(AC)}^2 - \mathcal{N}_{BA}^2 - \mathcal{N}_{BC}^2$, and

$\pi_C = \mathcal{N}_{C(AB)}^2 - \mathcal{N}_{CA}^2 - \mathcal{N}_{CB}^2$. However, they indicated that the residual entanglement corresponding to the different focus varies under permutations of the qubits, i.e., generally $\pi_A \neq \pi_B \neq \pi_C$.

Entanglement monotone is an important quality for entanglement measures. Any increase in correlations achieved by LOCC should be naturally classical. In other words, entanglement should be non-increasing under LOCC. Therefore monotonicity for entanglement measure under LOCC is considered as the natural requirement [8]. The symmetry of entanglement measure under permutations implies that the measure represents a collective property of the qubits which is unchanged by permutations [7]. In this paper, we present entanglement measures for n qubits, and demonstrate that the entanglement measures in question are (i) entanglement monotones, i.e., non-increasing on average under LOCC in all the n qubits, (ii) invariant under permutations of the qubits, and (iii) multiplicative in some cases.

In this paper, in Sec. 2 we study the entanglement measure $\tau(\psi)$ for even n qubits. In Sec. 3, we investigate the residual entanglement $\tau^{(i)}(\psi)$ with respect to qubit i and the odd n -tangle $R(\psi)$ for odd n qubits. $\tau(\psi)$, $\tau^{(i)}(\psi)$, and $R(\psi)$ only require $+$, $-$, and $*$ operations.

Notations: (1). Let $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i|i\rangle$ and $|\psi'\rangle = \sum_{i=0}^{2^n-1} a'_i|i\rangle$ be states of n qubits in this paper.

(2). Let $i_{n-1}...i_1i_0$ be an n -bit binary representation of i . That is, $i = i_{n-1}2^{n-1} + \dots + i_12^1 + i_02^0$.

Then, let $N(i)$ be the number of the occurrences of “1” in $i_{n-1}...i_1i_0$ and $N^*(i)$ be the number of the occurrences of “1” in $i_{n-2}...i_1i_0$, respectively.

2 Entanglement measure for even n qubits

In our previous work[11], we defined the entanglement measure of the state $|\psi\rangle$ of even n qubits as

$$\tau(\psi) = 2 |\mathcal{I}^*(a, n)|, \quad (2.1)$$

where

$$\mathcal{I}^*(a, n) = \sum_{i=0}^{2^{n-2}-1} \text{sgn}^*(n, i) (a_{2i}a_{(2^n-1)-2i} - a_{2i+1}a_{(2^n-2)-2i}). \quad (2.2)$$

The functions sgn and sgn^* have been defined previously in [11]. To facilitate reading, we have listed the definitions of sgn and sgn^* in Appendix A. When $n = 2$, $\tau(\psi) = 2 |a_0a_3 - a_1a_2|$, which is just the

concurrence for two qubits.

Theorem 1 in [11] implies that $\mathcal{I}^*(a, n)$ and $\tau(\psi)$ for even n qubits are invariant under SL (determinant-one) operators, especially under LU (local unitary) operators. In order to argue below that $\tau(\psi)$ for even n qubits is an entanglement monotone, we need the following result. If the states $|\psi'\rangle$ and $|\psi\rangle$ are related by a local operator as

$$|\psi'\rangle = \underbrace{\alpha \otimes \beta \otimes \gamma \cdots}_n |\psi\rangle, \quad (2.3)$$

then

$$\mathcal{I}^*(a', n) = \mathcal{I}^*(a, n) \underbrace{\det(\alpha) \det(\beta) \det(\gamma) \cdots}_n \quad (2.4)$$

and

$$\tau(\psi') = \tau(\psi) \underbrace{|\det(\alpha) \det(\beta) \det(\gamma) \cdots|}_n. \quad (2.5)$$

It is easy to see that Eq. (2.5) follows Eqs. (2.1) and (2.4). The proof of Eq. (2.4) is found in part A of Appendix D in [15] in which the condition that α, β, \dots are invertible was not used. Following this result, we have the following two results. (1). That the states $|\psi'\rangle$ and $|\psi\rangle$ are connected by SLOCC, i.e., α, β, \dots are invertible, becomes a special case of Eq. (2.3). Hence Eq. (2.5) holds. (2). Eq. (2.5) is true even if the states $|\psi'\rangle$ and $|\psi\rangle$ are connected by general LOCC, i.e., by non-invertible operators (see [9]).

2.1 Invariance under permutations of the n qubits

For a state of even n qubits, $|\psi\rangle$, we show in this section the invariance of $\tau(\psi)$ under permutations of the qubits. To this end, we first prove following propositions.

Remark 2.1 Each term of $\mathcal{I}^*(a, n)$ in Eq. (2.2) takes the form $(-1)^{N(k)} a_k a_{2^n-1-k}$.

Proof It is easy to see that binary representations of k and 2^n-1-k are complementary. So, $N(k) + N(2^n-1-k) = n$. Hence, $(-1)^{N(k)} = (-1)^{N(2^n-1-k)}$. By the definition for sgn^* in Appendix A, $sgn^*(n, i) = (-1)^{N(i)}$ when $0 \leq i \leq 2^{n-3}-1$ and $sgn^*(n, i) = (-1)^{n+N(i)}$ when $2^{n-3} \leq i \leq 2^{n-2}-1$. Therefore, $sgn^*(n, i) = (-1)^{N(i)}$ when n is even and $0 \leq i \leq 2^{n-2}-1$. Next there are two cases.

1. Consider term $\text{sgn}^*(n, i)a_{2i}a_{(2^n-1)-2i}$. Since $N(2i) = N(i)$, this remark is true for case 1.
2. Consider term $-\text{sgn}^*(n, i)a_{2i+1}a_{(2^n-2)-2i}$. Since $N(2i+1) = N(i) + 1$, this remark is true for case 2.

Lemma 2.2 *The term $\mathcal{I}^*(a, n)$ in Eq. (2.2) does not vary under any permutation of the n qubits.*

Proof By remark 2.1, each term of $\mathcal{I}^*(a, n)$ is of the form $(-1)^{N(k)}a_k a_{2^n-1-k}$. Let the binary number for k correspond to the binary number for k' under permutation π of the qubits. Then, the binary number for $2^n - 1 - k$ corresponds to the binary number for $2^n - 1 - k'$ under π . That is, $\pi(2^n - 1 - k) = 2^n - 1 - k'$. Obviously, $a_k = a_{k'}$, $a_{2^n-1-k} = a_{2^n-1-k'}$, and $N(k) = N(k')$. Thus, $(-1)^{N(k)}a_k a_{2^n-1-k} = (-1)^{N(k')}a_{k'} a_{2^n-1-k'}$. Therefore, $\mathcal{I}^*(a, n)$ does not vary under any permutation of the qubits.

From lemma 2.2 and Eq. (2.1), we have the following corollary 1.

Corollary 2.3 *The residual entanglement $\tau(\psi)$ does not vary under any permutation of the n qubits.*

2.2 Product states

For product states, the residual entanglement $\tau(\psi)$ either vanishes or is multiplicative. In this section, we state an important theorem and refer the reader to the Appendix B for a detailed proof.

Theorem 2.4 *Let $|\psi\rangle$ be a state of even n qubits which can be expressed as a tensor product state of state $|\phi\rangle$ of the first l qubits and state $|\omega\rangle$ of the rest $(n-l)$ qubits. Let $|\phi\rangle = \sum_{i=0}^{2^l-1} b_i|i\rangle$, where $1 \leq l < n$, and $|\omega\rangle = \sum_{i=0}^{2^{n-l}-1} c_i|i\rangle$. Then, $\tau(\psi) = \tau(\phi)\tau(\omega)$ for even l while $\tau(\psi) = 0$ for odd l .*

Proof See Appendix B for a detailed proof.

It is instructive to look at several examples to see the usefulness of this theorem. In example 1, we show a four-qubit state in which $\tau(\psi) = 1$ and in example 2, we look at a case of a six-qubit state in which $\tau(\psi) = 0$.

Example For four qubits, $\tau((1/2)(|00\rangle + |11\rangle)_{12} \otimes (|00\rangle + |11\rangle)_{34}) = 1$.

Example For six qubits, $\tau((1/2)((|000\rangle + |111\rangle)_{123} \otimes (|000\rangle + |111\rangle)_{456})) = 0$.

It is possible to extend theorem 1 further. From theorem 1 and corollary 1, we have the following corollary 2.5:

Corollary 2.5 *(An extension of theorem 1): (1). If $|\psi\rangle$ is a tensor product state of state $|\phi\rangle$ of even qubits and state $|\omega\rangle$ of even qubits, then $\tau(\psi) = \tau(\phi)\tau(\omega)$. That is, $\tau(\psi)$ is multiplicative. (2). If $|\psi\rangle$ is a tensor product state of state $|\phi\rangle$ of odd qubits and state $|\omega\rangle$ of odd qubits, then $\tau(\psi) = 0$.*

The corollary 2.5 argues that $\tau(\psi)$ for even n qubits is not a measure of n -way entanglement. Note that the conjecture for even n qubits in [11] is the same as Corollary 2. At this juncture, it is probably interesting to note some examples for six-qubit states.

Example For six qubits, $\tau((1/2)((|0000\rangle + |1111\rangle)_{1456} \otimes (|00\rangle + |11\rangle)_{23})) = 1$.

Example For six qubits, $\tau((1/2)((|000\rangle + |111\rangle)_{135} \otimes (|000\rangle + |111\rangle)_{246})) = 0$.

In [9] SLOCC classes of three qubits are related by means of non-invertible operators, i.e., of general LOCC, see Fig.1 in [9]. Unfortunately, we can not derive a nice result for four qubits. For example, for four qubits, no non-invertible operators can transform the state $|GHZ\rangle$ to a state within $|GHZ\rangle_{12} \otimes |GHZ\rangle_{34}$ SLOCC class. Assume that the states $|\phi\rangle$ and $|GHZ\rangle$ are connected by a non-invertible operator as $|\phi\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta |GHZ\rangle$. Then by Eq. (2.5), $\tau(\phi) = \tau(GHZ)|\det(\alpha)\det(\beta)\det(\gamma)\det(\delta)| = 0$. However, for any state $|\phi\rangle$ in $|GHZ\rangle_{12} \otimes |GHZ\rangle_{34}$ SLOCC class, $\tau(\phi) \neq 0$ by Eq. (2.5) and Example 1.

2.3 Entanglement monotone

As indicated in [8], a natural measure of entanglement should also be an entanglement monotone. Let us follow the idea in [9] to prove that $\tau(\psi)$ for n qubits is an entanglement monotone. Based on the work in [9], it is enough to consider two-outcome POVM's and apply POVM's to one party. For example, we can simply apply a local POVM to qubit k . Let A_1 and A_2 be the two POVM elements such that $A_1^\dagger A_1 + A_2^\dagger A_2 = I$. By the singular value decomposition, there are unitary matrices U_i and V_i and diagonal matrices D_i with

non-negative entries such that $A_i = U_i D_i V_i$ [2], where $D_1 = \text{diag}(a, b)$ and $D_2 = \text{diag}((1-a^2)^{1/2}, (1-b^2)^{1/2})$ [9]. Let $|\psi\rangle$ be an initial state and

$$|\bar{\phi}_i\rangle = \underbrace{I \otimes \dots \otimes I}_{k-1} \otimes A_i \otimes \underbrace{I \otimes \dots \otimes I}_{n-k} |\psi\rangle \quad (2.6)$$

be the states after the application of the POVM for any n qubits, where I is an identity. To normalize $|\bar{\phi}_i\rangle$, let $|\phi_i\rangle = |\bar{\phi}_i\rangle/\sqrt{p_i}$, where $p_i = \langle \bar{\phi}_i | \bar{\phi}_i \rangle$. Clearly $p_1 + p_2 = 1$ [2]. As discussed in [9], next we can consider

$$\langle \tau^\eta \rangle = p_1 \tau^\eta(\phi_1) + p_2 \tau^\eta(\phi_2), \text{ where } 0 < \eta \leq 1, \quad (2.7)$$

and prove

$$\langle \tau^\eta \rangle \leq \tau^\eta(\psi) \quad (2.8)$$

to show that τ is an entanglement monotone.

It is intuitive that $\tau(\phi_i) = \tau(\bar{\phi}_i)/p_i$ because τ is a quadratic function with respect to its coefficients in the standard basis, see Eq. (2.2). Note that τ is a quartic function in [9][10]. By Eqs. (2.5) and (2.6),

$$\tau(\bar{\phi}_i) = \tau(\psi) |\det(A_i)| = \tau(\psi) |\det(D_i)|. \quad (2.9)$$

So, it is trivial to get $\tau(\bar{\phi}_1) = ab\tau(\psi)$ and $\tau(\bar{\phi}_2) = [(1-a^2)(1-b^2)]^{1/2}\tau(\psi)$. By substituting $\tau(\bar{\phi}_1)$ and $\tau(\bar{\phi}_2)$ into Eq. (2.7), we get

$$\langle \tau^\eta \rangle = \{p_1 \frac{(ab)^\eta}{p_1^\eta} + p_2 \frac{[(1-a^2)(1-b^2)]^{\eta/2}}{p_2^\eta}\} \tau^\eta(\psi). \quad (2.10)$$

When $\eta = 1$,

$$\langle \tau \rangle = \{ab + [(1-a^2)(1-b^2)]^{1/2}\} \tau(\psi). \quad (2.11)$$

As discussed in [9], it is easy to derive $\langle \tau \rangle \leq \tau(\psi)$. Thus, this means when $\eta = 1$, τ is an entanglement monotone. Finally, as pointed out in [9], when $0 < \eta \leq 1$, it is easy to show that τ is an entanglement monotone.

It is worthwhile pointing out that in [9] the authors simplified the calculation for $\tau(\bar{\phi}_i)$ [9][10] by using the restriction $V_1 = V_2$ since they apparently thought the fact that A_1 and A_2 constitute a POVM implies $V_1 = V_2$. The authors in [9] and [10] used the invariance of the 3-tangle in [9] and the even n -tangle in [10]

under permutations of the qubits, respectively, to consider a local POVM in party A only. Moreover, they also used the invariance of the 3-tangle and the even n -tangle under LU respectively to obtain $\tau(U_i D_i V \psi) = \tau(D_i V \psi)$ in [9][10].

3 Residual entanglement for odd n qubits and the odd n -tangle

In this section, we propose the residual entanglement with respect to each qubit. We consider $\tau(\psi)$ for odd n qubits in [11] as the residual entanglement with respect to qubit 1. Let $(1, i)$ be the transposition of qubits 1 and i , and $(1, i)|\psi\rangle$ be the state obtained from $|\psi\rangle$ under the transposition $(1, i)$. Let $\tau^{(i)}(\psi) = \tau((1, i)\psi)$, $i = 2, 3, \dots, n$ and $\tau^{(1)}(\psi) = \tau(\psi)$. Then, we propose $\tau^{(i)}(\psi)$ as the residual entanglement with respect to qubit i , where $i = 1, \dots, n$. It seems that the residual entanglement $\tau^{(1)}(\psi)$ with respect to qubit 1 is transferred to qubit i under the transposition $(1, i)$. By averaging the residual entanglement with respect to each qubit, we propose the following $R(\psi)$ as the odd n -tangle.

$$R(\psi) = \frac{1}{n} \sum_{i=1}^n \tau^{(i)}(\psi). \quad (3.1)$$

First, we study the properties of $\tau(\psi)$. Then, by means of the properties of $\tau(\psi)$, we investigate the residual entanglement $\tau^{(i)}(\psi)$ with respect to qubit i and the odd n -tangle $R(\psi)$.

In [11], we defined the entanglement measure for the state $|\psi\rangle$ of odd n qubits as

$$\tau(\psi) = 4|(\overline{\mathcal{I}}(a, n))^2 - 4\mathcal{I}^*(a, n-1)\mathcal{I}_{+2^{n-1}}^*(a, n-1)|, \quad (3.2)$$

where

$$\begin{aligned} \overline{\mathcal{I}}(a, n) = & \sum_{i=0}^{2^{n-3}-1} \text{sgn}(n, i) [(a_{2i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}) \\ & - (a_{(2^{n-1}-2)-2i} a_{(2^{n-1}+1)+2i} - a_{(2^{n-1}-1)-2i} a_{2^{n-1}+2i})], \end{aligned} \quad (3.3)$$

$$\begin{aligned}\mathcal{I}_{+2^{n-1}}^*(a, n-1) &= \sum_{i=0}^{2^{n-3}-1} \text{sgn}^*(n-1, i) \times \\ &\quad (a_{2^{n-1}+2i} a_{(2^n-1)-2i} - a_{2^{n-1}+1+2i} a_{(2^n-2)-2i}),\end{aligned}\tag{3.4}$$

$$\begin{aligned}\mathcal{I}^*(a, n-1) &= \sum_{i=0}^{2^{n-3}-1} \text{sgn}^*(n-1, i) \times \\ &\quad (a_{2i} a_{(2^{n-1}-1)-2i} - a_{2i+1} a_{(2^{n-1}-2)-2i}).\end{aligned}\tag{3.5}$$

For $n = 3$, $\tau(\psi)$ in Eq. (3.2) is just simply the residual entanglement for three qubits [9], i.e., 3 tangle, which is $\tau_{ABC} = 4|d_1 - 2d_2 + 4d_3|$, where the expressions for d_i are omitted here.

Theorem 2 in [11] implies that $(\bar{\mathcal{I}}(a, n))^2 - 4\mathcal{I}^*(a, n-1)\mathcal{I}_{+2^{n-1}}^*(a, n-1)$ and $\tau(\psi)$ are invariant under SL -operators, especially under LU -operators. We argue below that the entanglement measure τ for odd n qubits is an entanglement monotone, using the following result.

If the states $|\psi'\rangle$ and $|\psi\rangle$ are connected by a local operator as

$$|\psi'\rangle = \underbrace{\alpha \otimes \beta \otimes \gamma \cdots}_n |\psi\rangle,\tag{3.6}$$

then

$$\begin{aligned}(\bar{IV}(a', n))^2 - 4IV^*(a', n-1)IV_{+2^{n-1}}^*(a', n-1) &= \\ [(\bar{IV}(a, n))^2 - 4IV^*(a, n-1)IV_{+2^{n-1}}^*(a, n-1)] \times \\ \underbrace{(\det(\alpha) \det(\beta) \det(\gamma) \dots)^2}_n,\end{aligned}\tag{3.7}$$

and

$$\tau(\psi') = \tau(\psi) \underbrace{|\det(\alpha) \det(\beta) \det(\gamma) \dots|}_n^2.\tag{3.8}$$

It is easy to know that Eq. (3.8) follows Eqs. (3.2) and (3.7). For the proof of Eq. (3.7), see the proof in part B of Appendix D in [15] in which the condition that α, β, \dots are invertible was not used. Following

this result, for odd n qubits we also have the following two results. (1). That the states $|\psi'\rangle$ and $|\psi\rangle$ are connected by SLOCC, i.e., α, β, \dots are invertible, becomes a special case of Eq. (3.6). Hence Eq. (3.8) holds. (2). Eq. (3.8) is true even if the states $|\psi'\rangle$ and $|\psi\rangle$ are connected by general LOCC, i.e., by non-invertible operators (see [9]).

3.1 Invariance of $\tau(\psi)$ under any permutation of the qubits 2, 3, ..., n .

The residual entanglement $\tau(\psi)$ is invariant under permutation of these qubits. To prove the invariance, we prove the following remark 3.1 and lemma 3.2, and corollary 3.3 stated below.

Remark 3.1 *Let $|\psi\rangle$ be a state of odd n qubits. Then each term of $\overline{\mathcal{I}}(a, n)$ in Eq. (3.3) is of the form $(-1)^{N^*(k)} a_k a_{2^n-1-k}$.*

Proof Since the binary representations of k and $2^n - 1 - k$ are complementary, $N(k) + N(2^n - 1 - k) = n$ and $N^*(k) + N^*(2^n - 1 - k) = n - 1$. Hence, $(-1)^{N^*(k)} = (-1)^{N^*(2^n-1-k)}$. Note that $\text{sgn}(n, i) = (-1)^{N(i)}$ when $0 \leq i \leq 2^{n-3} - 1$ by the definition for sgn^* in Appendix A. Next there are four cases.

1. Term $\text{sgn}(n, i) a_{2i} a_{(2^n-1)-2i}$. Since $0 \leq 2i \leq 2^{n-2} - 2$, $N^*(2i) = N(2i) = N(i)$.
2. Term $-\text{sgn}(n, i) a_{2i+1} a_{(2^n-2)-2i}$. Since $1 \leq 2i+1 \leq 2^{n-2} - 1$, $N^*(2i+1) = N(2i+1) = N(i) + 1$.
3. Term $-\text{sgn}(n, i) a_{(2^{n-1}-2)-2i} a_{(2^{n-1}+1)+2i}$. Clearly, $N^*(2^{n-1} + 1 + 2i) = N^*(1 + 2i) = N(i) + 1$.
4. Term $\text{sgn}(n, i) a_{(2^{n-1}-1)-2i} a_{2^{n-1}+2i}$. It is trivial that $N^*(2^{n-1} + 2i) = N(2i) = N(i)$.

Since the above four cases exhaust all possibilities, the remark holds.

Lemma 3.2 *Let $|\psi\rangle$ be a state of odd n qubits. Then, $\overline{\mathcal{I}}(a, n)$ in Eq. (3.3) does not vary under any permutation of the qubits 2, 3, ..., and n .*

Proof By remark 3.1, each term of $\overline{\mathcal{I}}(a, n)$ in Eq. (3.3) is of the form $(-1)^{N^*(k)} a_k a_{2^n-1-k}$. Let the binary number for k correspond to the binary number for k' under permutation π of the qubits 2, 3, ..., and n . Then, the binary number for $2^n - 1 - k$ corresponds to the binary number for $2^n - 1 - k'$ under π . That is, $\pi(2^n - 1 - k) = 2^n - 1 - k'$. Obviously, $a_k = a_{k'}$, $a_{2^n-1-k} = a_{2^n-1-k'}$, and $N^*(k) = N^*(k')$. Thus, $(-1)^{N^*(k)} a_k a_{2^n-1-k} = (-1)^{N^*(k')} a_{k'} a_{2^n-1-k'}$. Therefore, $\overline{\mathcal{I}}(a, n)$ does not vary under any permutation of the qubits 2, 3, ..., and n .

Finally, we have the following corollary concerning the invariance of the entanglement measure $\tau(\psi)$ under any permutations of the qubits 2, 3, ..., and n .

Corollary 3.3 *Let $|\psi\rangle$ be a state of odd n qubits. Then, $\tau(\psi)$ does not vary under any permutation of the qubits 2, 3, ..., and n .*

Proof Note that a binary representation of each subscript in each term of $\mathcal{I}^*(a, n-1)$ in Eq. (3.5) is of the form $0k_{n-2}...k_1k_0$ and a binary representation of each subscript in each term of $\mathcal{I}_{+2^{n-1}}^*(a, n-1)$ in Eq. (3.4) is of the form $1k_{n-2}...k_1k_0$. Under any permutation of the qubits 2, 3, ..., and n , by lemma 2.2 either $\mathcal{I}^*(a, n-1)$ or $\mathcal{I}_{+2^{n-1}}^*(a, n-1)$ does not vary and by lemma 3.2 $\overline{\mathcal{I}}(a, n)$ does not vary. Hence, by the definition in Eq. (3.2), $\tau(\psi)$ does not vary under any permutation of the qubits 2, 3, ..., and n .

To see the usefulness of the results that we have shown, it is instructive to study an example:

Example Let $|\psi\rangle = (1/2)((|00\rangle + |11\rangle)_{12} \otimes (|000\rangle + |111\rangle)_{345})$. Then, by Eq. (3.2), a simple calculation shows that $\tau(\psi) = 0$. Under the transposition (1, 5) of the qubits 1 and 5, $|\psi\rangle$ becomes $|\psi'\rangle = (1/2)((|00\rangle + |11\rangle)_{25} \otimes (|000\rangle + |111\rangle)_{134})$. By Eq. (3.2), $\tau(\psi') = 1$.

3.2 Product states

For product states, $\tau(\psi)$ vanishes or is multiplicative. To prove this statement, we have the following theorem:

Theorem 3.4 *Let $|\psi\rangle$ be a state of odd n qubits and a tensor product state of the state $|\phi\rangle$ of the first l qubits and the state $|\omega\rangle$ of the rest $(n-l)$ qubits. Let $|\phi\rangle = \sum_{i=0}^{2^l-1} b_i|i\rangle$, where $1 \leq l < n$, and $|\omega\rangle = \sum_{i=0}^{2^{n-l}-1} c_i|i\rangle$. Then, $\tau(\psi) = \tau(\phi)\tau^2(\omega)$ for odd l , while $\tau(\psi) = 0$ for even l .*

Proof See Appendix C for the detailed proof.

It is interesting to study some examples to see some application of the theorem.

Example For five qubits, $\tau((1/2)(|000\rangle + |111\rangle)_{123} \otimes (|00\rangle + |11\rangle)_{45}) = 1$.

Example For five qubits, $\tau((1/2)(|00\rangle + |11\rangle)_{12} \otimes (|000\rangle + |111\rangle)_{345}) = 0$.

Moreover, from theorem 3.4 and corollary 3.3, we have the following corollary as an extension of theorem 3.4.

Corollary 3.5 *Theorem 3.4 holds under any permutation π of the qubits $2, 3, \dots$, and n . That is, let $|\phi\rangle$ be a state of l qubits including qubit 1 and state $|\omega\rangle$ be a state of the rest $(n-l)$ qubits, then, $\tau(\psi) = \tau(\phi)\tau^2(\omega)$ for odd l while $\tau(\psi) = 0$ for even l . Hence, $\tau(\psi)$ can be considered to be multiplicative for odd l .*

The corollary 3.5 implies that for odd n qubits, $\tau(\psi)$ is not a measure of n -way entanglement. In [11], we conjectured that $\tau(\psi) = 0$ whenever ψ is a product states of the odd n qubits. This corollary indicates that the conjecture is not always true.

Example For five qubits, $\tau((1/2)(|000\rangle + |111\rangle)_{125} \otimes (|00\rangle + |11\rangle)_{34}) = 1$ and $\tau((1/2)(|00\rangle + |11\rangle)_{15} \otimes (|000\rangle + |111\rangle)_{234}) = 0$.

For five qubits, by resorting to the iterative formula about the number of the degenerate SLOCC classes in [16], there are $5 \times t(4) + 66$ degenerate SLOCC classes, where $t(4)$ is the number of true SLOCC entanglement classes for four qubits. In [16], 28 true SLOCC classes for four qubits were found. Hence, in total, there are at least 206 degenerate SLOCC classes for five qubits. Note that degenerate SLOCC classes are SLOCC classes of product states.

By corollary 3.5, for five qubits, τ always vanishes for all the product states except for the states within the following SLOCC classes:

$$\begin{aligned} &(|000\rangle + |111\rangle)_{123} \otimes (|00\rangle + |11\rangle)_{45}, (|000\rangle + |111\rangle)_{124} \otimes (|00\rangle + |11\rangle)_{35}, \\ &(|000\rangle + |111\rangle)_{125} \otimes (|00\rangle + |11\rangle)_{34}, (|000\rangle + |111\rangle)_{134} \otimes (|00\rangle + |11\rangle)_{25}, \\ &(|000\rangle + |111\rangle)_{135} \otimes (|00\rangle + |11\rangle)_{24}, (|000\rangle + |111\rangle)_{145} \otimes (|00\rangle + |11\rangle)_{23}. \end{aligned}$$

As discussed in [9], SLOCC classes of three qubits are related by means of non-invertible operators, i.e., of general LOCC, see Fig.1 in [9]. Here, we want to show that it is not true for five qubits. For example, no non-invertible operators can transform the state $|GHZ\rangle$ to a state within $|GHZ\rangle_{123} \otimes |GHZ\rangle_{45}$ SLOCC class. Assume that the states $|\phi\rangle$ and $|GHZ\rangle$ are connected by a non-invertible operator as $|\phi\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \eta |GHZ\rangle$. Then by Eq. (3.8), $\tau(\phi) = \tau(\psi) |\det^2(\alpha) \det^2(\beta) \det^2(\gamma) \det^2(\delta) \det^2(\eta)| = 0$. However, for any state $|\phi\rangle$ in $|GHZ\rangle_{123} \otimes |GHZ\rangle_{45}$ SLOCC class, $\tau(\phi) \neq 0$ [11].

3.3 Entanglement monotone.

It is easy to see that the first paragraph of Sec. 2.3 is true for any n qubits. It is not hard to show that $\tau(\phi_i) = \tau(\bar{\phi}_i)/p_i^2$ because τ is a quartic function with respect to its coefficients in the standard basis, see Eqs. (3.2)-(3.5). By Eqs. (2.6) and (3.8),

$$\tau(\bar{\phi}_i) = \tau(\psi) |\det(A_i)|^2 = \tau(\psi) |\det(D_i)|^2. \quad (3.9)$$

So, $\tau(\bar{\phi}_1) = (ab)^2 \tau(\psi)$ and $\tau(\bar{\phi}_2) = (1 - a^2)(1 - b^2) \tau(\psi)$. By substituting $\tau(\bar{\phi}_1)$ and $\tau(\bar{\phi}_2)$ into Eq. (2.7), we get

$$\langle \tau^\eta \rangle = \left\{ p_1 \frac{(ab)^{2\eta}}{p_1^{2\eta}} + p_2 \frac{[(1 - a^2)(1 - b^2)]^\eta}{p_2^{2\eta}} \right\} \tau^\eta(\psi). \quad (3.10)$$

Eq. (3.10) was also obtained in [9]. Therefore the rest of the proof is the same as the one in [9].

Note that in the above proof, we do not use the restriction $V_1 = V_2$, the invariance of τ under permutations of the qubits, or the invariance of τ under LU . Therefore, it is not necessary to establish a relation between the invariance of a measure under permutations of the qubits and an entanglement monotone.

3.4 The residual entanglement with respect to each qubit and the odd n -tangle

3.4.1 The residual entanglement $\tau^{(i)}(\psi)$ with respect to qubit i

It is plain to derive that $\tau^{(i)}(\psi)$ satisfy Eq. (3.8). From the properties of $\tau(\psi)$, one can obtain that (1). $0 \leq \tau^{(i)}(\psi) \leq 1$; (2). $\tau^{(i)}(\psi)$ are SL -invariant, especially LU -invariant; (3). $\tau^{(i)}(\psi)$ are entanglement monotones. (4). $\tau^{(i)}(\psi)$, $i = 1, 2, \dots, n$, are invariant under permutations of the qubits: $1, \dots, (i-1)$, $(i+1), \dots, n$; (5). When $|\psi\rangle$ is a product state of odd n qubits, that is, $|\psi\rangle = |\phi\rangle \otimes |\omega\rangle$, where $|\phi\rangle$ is a state of l qubits including qubit i , and $|\omega\rangle$ is a state of $(n-l)$ qubits, then $\tau^{(i)}(\psi) = \tau^{(i)}(\phi)(\tau^{(i)}(\omega))^2$ for odd l while $\tau^{(i)}(\psi) = 0$ for even l .

We argue that the above (5) holds as follows. Let $(1, i)$ be a transposition of qubits 1 and i , and the state $(1, i)|\psi\rangle$ be obtained from $|\psi\rangle$ under the transposition $(1, i)$. It is not hard to see that $\tau^{(i)}(\psi) = \tau((1, i)|\psi\rangle) = \tau(((1, i)|\phi\rangle \otimes (1, i)|\omega\rangle))$. There are two cases. Case 1. In this case, qubit 1 occurs in $|\phi\rangle$. Under the transposition $(1, i)$, qubits 1 and i occur in $(1, i)|\phi\rangle$, and $(1, i)|\omega\rangle = |\omega\rangle$. Case 2. In this case, qubit 1 occurs in $|\omega\rangle$. Under the transposition $(1, i)$, qubit 1 occurs in $(1, i)|\phi\rangle$ while qubit i occurs in $(1, i)|\omega\rangle$. In either case, by corollary 4, $\tau(((1, i)|\phi\rangle \otimes (1, i)|\omega\rangle)) = \tau((1, i)|\phi\rangle)\tau^2((1, i)|\omega\rangle) = \tau^{(i)}(\phi)(\tau^{(i)}(\omega))^2$ for odd l while $\tau(((1, i)|\phi\rangle \otimes (1, i)|\omega\rangle)) = 0$ for even l . For the proofs of (1), (2), (3), and (4), see [17].

3.4.2 The odd n -tangle

It is not difficult to show that $R(\psi)$ in Eq. (3.1) satisfies Eq. (3.8). Thus, from the properties of $\tau^{(i)}(\psi)$, one can derive that (1). $0 \leq R \leq 1$; (2). R is invariant under SL -operators, especially LU -operators; (3). R is an entanglement monotone; (4). $R(\psi)$ is invariant under any permutation of all the odd n qubits. For the proofs of (1), (2), (3), and (4), see [17]. However, $R(\psi)$ is not multiplicative.

Next let us see the performance of $R(\psi)$ for three qubits. Let $n = 3$. As discussed before, $\tau(\psi)$ happens to be Coffman et al.'s residual entanglement for three qubits. From (5) of p. 429 in [18], $\tau(\psi) = \tau^{(1)}(\psi) = \tau^{(2)}(\psi) = \tau^{(3)}(\psi)$. Thus, $R(\psi) = \tau(\psi)$. That is, $R(\psi)$ is just Coffman et al.'s residual entanglement for three qubits.

4 Summary

We summarize this paper as follows. We demonstrate that the entanglement measure $\tau(\psi)$ for even n qubits, the residual entanglement $\tau^{(i)}(\psi)$ with respect to qubit i and the odd n -tangle $R(\psi)$ for odd n qubits satisfy the following properties. (1). $\tau(\psi)$, $\tau^{(i)}(\psi)$, and $R(\psi)$ are between 0 and 1; (2). $\tau(\psi)$, $\tau^{(i)}(\psi)$, and $R(\psi)$ are SL -invariant, especially LU -invariant; (3). $\tau(\psi)$, $\tau^{(i)}(\psi)$, and $R(\psi)$ are entanglement monotones; (4). $\tau(\psi)$ for even n qubits and the odd n -tangle $R(\psi)$ are invariant under permutations of all the qubits; however $\tau^{(i)}(\psi)$ are invariant only under permutations of the qubits: $1, \dots, (i-1), (i+1), \dots, n$. (5). For product states, i.e., $|\psi\rangle = |\phi\rangle \otimes |\omega\rangle$, for even n qubits, if $|\phi\rangle$ is a state of even qubits then $\tau(\psi) = \tau(\phi)\tau(\omega)$ else $\tau(\psi) = 0$; for odd n qubits, if $|\phi\rangle$ is a state of l qubits including qubit i , then $\tau^{(i)}(\psi) = \tau^{(i)}(\phi)(\tau^{(i)}(\omega))^2$ for odd l while $\tau(\psi) = 0$ for even l .

Monotonicity is a natural requirement for entanglement measure. The symmetry of entanglement measure under permutations represents a collective property of the qubits. Therefore the entanglement measures presented in this paper are natural.

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Appendix A. Properties of sgn and sgn^*

In order to show the invariance of the entanglement measure for even (odd) n qubits, we need the following properties of the function sgn (sgn^*). The functions sgn and sgn^* were recursively defined in [11]. For readability, we redefine sgn and sgn^* as follows.

Definition of sgn :

$$sgn(n, i) = (-1)^{N(i)} \text{ when } 0 \leq i \leq 2^{n-3} - 1. \quad (A1)$$

The definition of $N(i)$ is given in the last paragraph of this introduction. Whereas, $N(i)$ is the number

of the occurrences of “1” in the n -bit binary representation $i_{n-1}...i_1i_0$ of i .

In fact, this definition of $sgn(n, i)$ can be derived from the recursive definition of $sgn(n, i)$ in [11] by using the following property 1 about $N(i)$.

Definition of sgn^ :*

$$sgn^*(n, i) = \begin{cases} (-1)^{N(i)} & \text{for } 0 \leq i \leq 2^{n-3} - 1, \\ (-1)^{n+N(i)} & \text{for } 2^{n-3} \leq i \leq 2^{n-2} - 1. \end{cases}$$

This definition of $sgn^*(n, i)$ can be derived from the recursive definition of sgn^* in [11] by using the following property 1 about $N(i)$.

It is straightforward to derive the following property 1 about $N(i)$ by means of the definition of $N(i)$. The property 1 will be used in the proofs of the following properties 2-5.

Property 1:

- (i). Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then $N(k + j \times 2^{n-l-1}) = N(j) + N(k)$.
- (ii). Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq t \leq 2^{l-3} - 1$. Then $N(k + t2^{n-l}) = N(k) + N(t)$ and $N(k + (2t + 1)2^{n-l-1}) = N(k) + N(t) + 1$.
- (iii). Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then, $N(2^{n-l-1} - 1 - k) = n - l - 1 - N(k)$ and $N((j + 1) \times 2^{n-l-1} - 1 - k) = N(j) + n - l - 1 - N(k)$.

Proof Proof of (i):

Let the binary number of j be $j_{l-3}j_{l-4}...j_1j_0$, where $j_i \in \{0, 1\}$. That is, $j = j_{l-3} \times 2^{l-3} + ... + j_1 \times 2^1 + j_0 \times 2^0$. $j \times 2^{n-l-1} = j_{l-3} \times 2^{n-4} + ... + j_1 \times 2^{n-l} + j_0 \times 2^{n-l-1}$. Clearly, $N(j) = N(j \times 2^{n-l-1})$. Since $0 \leq k \leq 2^{n-l-2} - 1$, $N(k + j \times 2^{n-l-1}) = N(j \times 2^{n-l-1}) + N(k) = N(j) + N(k)$.

Proof of (ii):

Let the binary representation of t be $t_{l-4}...t_1t_0$ and the binary number of k be $k_{n-l-3}...k_1k_0$, where $t_i, k_i \in \{0, 1\}$. $k + (2t + 1)2^{n-l-1} = k + t2^{n-l} + 2^{n-l-1}$. The latter can be rewritten as $t_{l-4}2^{n-4} + ... + t_12^{n-l+1} + t_02^{n-l} + 2^{n-l-1} + k_{n-l-3}2^{n-l-3} + ... + k_02^0$. It is obvious that $N(k + t2^{n-l} + 2^{n-l-1}) = N(k) + N(t) + 1$. As well, $N(k + t2^{n-l}) = N(k) + N(t)$.

Proof of (iii):

Let us calculate $N(2^{n-l-1} - 1 - k)$. The binary number of $2^{n-l-1} - 1$ is $\underbrace{1 \dots 1}_{n-l-1}$. That is, $2^{n-l-1} - 1 = 2^{n-l-2} + \dots + 2^1 + 2^0$. Let $k_{n-l-3} \dots k_1 k_0$ be the binary number of k , where $k_i \in \{0, 1\}$. That is, $k = k_{n-l-3} \times 2^{n-l-3} + \dots + k_1 \times 2^1 + k_0 \times 2^0$. Note that the binary numbers of $2^{n-l-1} - 1 - k$ and k are complementary. Hence, it is straightforward that $N(2^{n-l-1} - 1 - k) = n - l - 1 - N(k)$.

$(j + 1) \times 2^{n-l-1} - 1 - k = j \times 2^{n-l-1} + (2^{n-l-1} - 1 - k)$. Notice that $2^{n-l-2} \leq 2^{n-l-1} - 1 - k \leq 2^{n-l-1} - 1$. It is intuitive that $N(j \times 2^{n-l-1} + (2^{n-l-1} - 1 - k)) = N(j) + N(2^{n-l-1} - 1 - k)$. Therefore, $N((j + 1) \times 2^{n-l-1} - 1 - k) = N(j) + n - l - 1 - N(k)$.

The following properties 2-5 are used in proofs of Theorems 1 and 2. The property 2 about sgn follows the property 1 and the definition for sgn .

Property 2:

Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then $sgn(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{n+l+1} sgn(n, k + j \times 2^{n-l-1})$.

Proof (1). Compute $sgn(n, k + j \times 2^{n-l-1})$. Since $k + j \times 2^{n-l-1} < 2^{n-3} - 1$, by the definition for sgn , $sgn(n, k + j \times 2^{n-l-1}) = (-1)^{N(k+j \times 2^{n-l-1})}$. By (i) of property 1, $N(k + j \times 2^{n-l-1}) = N(k) + N(j)$. Therefore $sgn(n, k + j \times 2^{n-l-1}) = (-1)^{N(j)+N(k)}$.

(2). Compute $sgn(n, (j + 1) \times 2^{n-l-1} - 1 - k)$. Since $(j + 1) \times 2^{n-l-1} - 1 - k \leq 2^{n-3} - 1$, by the definition for sgn , $sgn(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{N((j+1) \times 2^{n-l-1} - 1 - k)}$. By (iii) of property 1, $sgn(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{N(j)+n-l-1-N(k)}$.

Conclusively, $sgn(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{n+l+1} sgn(n, k + j \times 2^{n-l-1})$.

The property 3 about sgn and sgn^* can be shown by means of the property 1 and the definitions for sgn and sgn^* .

Property 3:

Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq t \leq 2^{l-3} - 1$. Then

(i) $sgn(n, k + (2t + 1) \times 2^{n-l-1}) = -sgn(n, k + t \times 2^{n-l})$.

$$(ii) \operatorname{sgn}^*(n-1, k + (2t+1) \times 2^{n-l-1}) = -\operatorname{sgn}^*(n-1, k + t \times 2^{n-l}).$$

Proof Proof of (i): Since $k + (2t+1) \times 2^{n-l-1} \leq 2^{n-3} - 2^{n-l-2} - 1$, by the definition for sgn , $\operatorname{sgn}(n, k + (2t+1) \times 2^{n-l-1}) = (-1)^{N(k+(2t+1) \times 2^{n-l-1})}$. By (ii) of property 1, $\operatorname{sgn}(n, k + (2t+1) \times 2^{n-l-1}) = (-1)^{N(k)+N(t)+1}$. Similarly, $\operatorname{sgn}(n, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}$.

Proof of (ii):

1. $0 \leq t \leq 2^{l-4} - 1$. Thus, $k + (2t+1) \times 2^{n-l-1} \leq 2^{n-4} - 1$ and $k + t \times 2^{n-l} \leq 2^{n-4} - 1$. By the definition for sgn^* and (ii) of property 1, $\operatorname{sgn}^*(n-1, k + (2t+1) \times 2^{n-l-1}) = (-1)^{N(k)+N(t)+1}$ and

$$\operatorname{sgn}^*(n-1, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}. \quad (\text{A2})$$

2. $2^{l-4} \leq t \leq 2^{l-3} - 1$. Thus, $2^{n-4} \leq k + t \times 2^{n-l} < 2^{n-3} - 1$ and $2^{n-4} < k + (2t+1) \times 2^{n-l-1} < 2^{n-3} - 1$. By the definition for sgn^* and (ii) of property 1, $\operatorname{sgn}^*(n-1, k + (2t+1) \times 2^{n-l-1}) = (-1)^{n-1}(-1)^{N(k)+N(t)+1}$ and

$$\operatorname{sgn}^*(n-1, k + t \times 2^{n-l}) = (-1)^{n-1}(-1)^{N(k)+N(t)}. \quad (\text{A3})$$

The property 4 about sgn^* can be obtained from the property 1 and the definition for sgn^* .

Property 4:

Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then $\operatorname{sgn}^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{n+l+1} \operatorname{sgn}^*(n-1, k + j \times 2^{n-l-1})$.

Proof 1. $0 \leq j \leq 2^{l-3} - 1$. Since $(j+1) \times 2^{n-l-1} - 1 - k \leq 2^{n-4} - 1$, by the definition for sgn^* , $\operatorname{sgn}^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{N((j+1) \times 2^{n-l-1} - 1 - k)}$. By (iii) of property 1, $\operatorname{sgn}^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{N(j)+n-l-1-N(k)}$. As well, since $k + j \times 2^{n-l-1} < 2^{n-4} - 2^{n-l-2} - 1$, by the definition for sgn^* , $\operatorname{sgn}^*(n-1, k + j \times 2^{n-l-1}) = (-1)^{N(k)+j \times 2^{n-l-1}}$. By (i) of property 1, $\operatorname{sgn}^*(n-1, k + j \times 2^{n-l-1}) = (-1)^{N(k)+N(j)}$. Therefore, the property holds for this case.

2. $2^{l-3} \leq j \leq 2^{l-2} - 1$. Thus, $2^{n-4} + 2^{n-l-2} \leq (j+1) \times 2^{n-l-1} - 1 - k \leq 2^{n-3} - 1$ and $2^{n-4} + 2^{n-l-2} - 1 \leq k + j \times 2^{n-l-1} \leq 2^{n-3} - 1$. By the definition for sgn^* and (iii) of property 1, $sgn^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{n-1}(-1)^{N((j)+n-l-1-N(k))}$. By the definition for sgn^* and (i) of property 1, $sgn^*(n-1, k + j \times 2^{n-l-1}) = (-1)^{n-1}(-1)^{N(k)+N(j)}$. Therefore, this property holds for this case.

It is not hard to derive the property 5 by means of the property 1 and the definitions for sgn and sgn^* .

Property 5:

Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq t \leq 2^{l-3} - 1$. When n is odd and l is odd or n is even and l is even, then the following statements are true:

- (i). $sgn^*(n-l, k) = (-1)^{N(k)}$,
- (ii). $sgn(n, k + t \times 2^{n-l}) = sgn^*(n-l, k)sgn(l, t)$,
- (iii). $sgn^*(n-1, k + t \times 2^{n-l}) = sgn^*(n-l, k)sgn^*(l-1, t)$.

Proof Proof of (i):

- 1. $0 \leq k \leq 2^{n-l-3} - 1$. By the definition for sgn^* , $sgn^*(n-l, k) = (-1)^{N(k)}$.
- 2. $2^{n-l-3} \leq k \leq 2^{n-l-2} - 1$. By the definition for sgn^* , $sgn^*(n-l, k) = (-1)^{n-l}(-1)^{N(k)}$. When n is odd and l is odd or n is even and l is even, clearly $sgn^*(n-l, k) = (-1)^{N(k)}$.

From cases 1 and 2, this statement follows.

Proof of (ii):

Step 1. Compute $sgn(n, k + t \times 2^{n-l})$. Since $0 \leq k + t \times 2^{n-l} \leq 2^{n-3} - 2^{n-l} + 2^{n-l-2} - 1$, by the definition for sgn and (ii) of property 1, $sgn(n, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}$.

Step 2. Compute $sgn(l, t)$. By the definition for sgn , $sgn(l, t) = (-1)^{N(t)}$. From (i) of this property and steps 1 and 2, we can conclude that (ii) holds.

Proof of (iii):

- 1. $0 \leq t \leq 2^{l-4} - 1$. By the definition for sgn^* , $sgn^*(l-1, t) = (-1)^{N(t)}$. By Eq. (A2), $sgn^*(n-1, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}$. Therefore, by (i) of this property, (iii) is true for this case.

2. $2^{l-4} \leq t \leq 2^{l-3} - 1$. By the definition for sgn^* , $sgn^*(l-1, t) = (-1)^{l-1}(-1)^{N(t)}$. By Eq. (A3), $sgn^*(n-1, k+t \times 2^{n-l}) = (-1)^{n-1}(-1)^{N(k)+N(t)}$. By (i) of this property, it is not hard to see that (iii) holds for this case.

Appendix B. The proof of Theorem 1

We show this theorem in three cases: case 1, $l = 1$; case 2, $l = 2$; case 3, $l \geq 3$.

Proof of $l = 1$:

Proof When $l = 1$, $|\phi\rangle = b_0|0\rangle + b_1|1\rangle$. By solving $|\psi\rangle = |\phi\rangle_1 \otimes |\omega\rangle_{2,\dots,n}$, we obtain the following amplitudes:

$$a_i = b_0 c_i, \quad a_{2^{n-1}+i} = b_1 c_i, \quad 0 \leq i \leq 2^{n-1} - 1. \quad (\text{B1})$$

By substituting the amplitudes in Eq. (B1) into $\mathcal{I}^*(a, n)$ in Eq. (2.2),

$$\begin{aligned} \mathcal{I}^*(a, n) &= b_0 b_1 \sum_{i=0}^{2^{n-2}-1} sgn^*(n, i) (c_{2i} c_{(2^{n-1}-1)-2i} - c_{2i+1} c_{(2^{n-1}-2)-2i}) \\ &= b_0 b_1 \sum_{i=0}^{2^{n-3}-1} sgn^*(n, i) (c_{2i} c_{(2^{n-1}-1)-2i} - c_{2i+1} c_{(2^{n-1}-2)-2i}) + \\ &\quad b_0 b_1 \sum_{i=2^{n-3}}^{2^{n-2}-1} sgn^*(n, i) (c_{2i} c_{(2^{n-1}-1)-2i} - c_{2i+1} c_{(2^{n-1}-2)-2i}). \end{aligned}$$

Let $k = 2^{n-2} - 1 - i$. Then the last sum can be rewritten as

$$-b_0 b_1 \sum_{k=2^{n-3}-1}^0 sgn^*(n, 2^{n-2} - 1 - k) (c_{2k} c_{(2^{n-1}-1)-2k} - c_{2k+1} c_{(2^{n-1}-2)-2k}).$$

It is easy to demonstrate $sgn^*(n, 2^{n-2} - 1 - k) = sgn^*(n, k)$ by the definition of sgn^* . Thus, $\mathcal{I}^*(a, n) = 0$ and $\tau(\psi) = 0$.

Proof of $l = 2$:

Proof In this case, $|\phi\rangle$ is a state of the first two qubits and $|\phi\rangle = \sum_{i=0}^3 b_i |i\rangle$, $|\omega\rangle$ is a state of the last $(n-2)$ -qubits and $|\omega\rangle = \sum_{i=0}^{2^{n-2}-1} c_i |i\rangle$. By the definition [11], $\tau(\phi) = 2|b_0 b_3 - b_1 b_2|$, $\tau(\omega) = 2|\mathcal{I}^*(c, n-2)|$, and $\tau(\psi) = 2|\mathcal{I}^*(a, n)|$. We can write

$$|\psi\rangle = |\phi\rangle_{1,2} \otimes |\omega\rangle_{3,\dots,n} \quad (\text{B2})$$

By solving Eq. (B2), we obtain the following amplitudes:

$$a_j = b_0 c_j, \quad a_{2^{n-2}+j} = b_1 c_j, \quad a_{2^{n-1}+j} = b_2 c_j, \quad a_{3 \times 2^{n-2}+j} = b_3 c_j \quad (\text{B3})$$

, where $0 \leq j \leq 2^{n-2} - 1$.

We rewrite $\mathcal{I}^*(a, n) = E_1 + E_2$, where

$$E_1 = \sum_{i=0}^{2^{n-3}-1} \text{sgn}^*(n, i) (a_{2i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}) \quad (\text{B4})$$

and

$$E_2 = \sum_{i=2^{n-3}}^{2^{n-2}-1} \text{sgn}^*(n, i) (a_{2i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}). \quad (\text{B5})$$

Let us compute E_1 as follows. Since $0 \leq i \leq 2^{n-3} - 1$, by Eq. (B3)

$$\begin{aligned} a_{2i} &= b_0 c_{2i}, \quad a_{(2^n-1)-2i} = b_3 c_{(2^{n-2}-1)-2i}, \\ a_{2i+1} &= b_0 c_{2i+1}, \quad a_{(2^n-2)-2i} = b_3 c_{(2^{n-2}-2)-2i}. \end{aligned} \quad (\text{B6})$$

By substituting the amplitudes in Eq. (B6) into E_1 , E_1 becomes

$$E_1 = b_0 b_3 \sum_{i=0}^{2^{n-3}-1} \text{sgn}^*(n, i) (c_{2i} c_{(2^{n-2}-1)-2i} - c_{2i+1} c_{(2^{n-2}-2)-2i}) \quad (\text{B7})$$

In Eq. (B7) let $E_1 = E_1^{(1)} + E_1^{(2)}$, where

$$E_1^{(1)} = b_0 b_3 \sum_{i=0}^{2^{n-4}-1} \text{sgn}^*(n, i) (c_{2i} c_{(2^{n-2}-1)-2i} - c_{2i+1} c_{(2^{n-2}-2)-2i}) \quad (\text{B8})$$

and

$$E_1^{(2)} = b_0 b_3 \sum_{i=2^{n-4}}^{2^{n-3}-1} \text{sgn}^*(n, i) (c_{2i} c_{(2^{n-2}-1)-2i} - c_{2i+1} c_{(2^{n-2}-2)-2i}). \quad (\text{B9})$$

Let us demonstrate $E_1^{(2)} = E_1^{(1)}$. Let $k = (2^{n-3} - 1) - i$. Then

$$E_1^{(2)} = -b_0 b_3 \sum_{k=2^{n-4}-1}^0 \text{sgn}^*(n, 2^{n-3} - 1 - k) (c_{2k} c_{(2^{n-2}-1)-2k} - c_{2k+1} c_{(2^{n-2}-2)-2k}). \quad (\text{B10})$$

When $0 \leq k \leq 2^{n-4} - 1$, by the definition for sgn^* and (iii) of property 1 in Appendix A, then $\text{sgn}^*(n, 2^{n-3} - 1 - k) = -\text{sgn}^*(n, k)$. Thus, $E_1^{(2)} = E_1^{(1)}$ and $E_1 = 2E_1^{(1)}$.

Next we show $E_1 = 2b_0b_3 \mathcal{I}^*(c, n-2)$. For this purpose, we only need to show $sgn^*(n, i) = sgn^*(n-2, i)$ provided that $0 \leq i \leq 2^{n-4} - 1$. The definition for sgn^* in Appendix A asserts this.

Similarly, we can derive $E_2 = -2b_1b_2 \mathcal{I}^*(c, n-2)$. Thus, $\mathcal{I}^*(a, n) = 2(b_0b_3 - b_1b_2) \mathcal{I}^*(c, n-2)$. Conclusively, $\tau(\psi) = \tau(\phi)\tau(\omega)$.

Proof for $l \geq 3$:

Proof We write

$$|\psi\rangle = |\phi\rangle_{1,\dots,l} \otimes |\omega\rangle_{(l+1),\dots,n}. \quad (\text{B11})$$

By solving equation Eq. (B11), we obtain the following amplitudes:

$$a_{k \times 2^{n-l} + i} = b_k c_i, k = 0, 1, \dots, (2^l - 1), i = 0, 1, \dots, (2^{n-l} - 1). \quad (\text{B12})$$

We rewrite $\mathcal{I}^*(a, n)$ as $\mathcal{I}^*(a, n) = \sum_{j=0}^{2^{l-1}-1} \Delta_j$, where

$$\begin{aligned} \Delta_j = & \sum_{i=j \times 2^{n-l-2}}^{(j+1) \times 2^{n-l-2}-1} sgn(n, i) [(a_{2i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}) \\ & + (a_{(2^{n-1}-2)-2i} a_{(2^{n-1}+1)+2i} - a_{(2^{n-1}-1)-2i} a_{2^{n-1}+2i})]. \end{aligned} \quad (\text{B13})$$

By substituting the amplitudes in Eq. (B12) into Δ_{2j} and Δ_{2j+1} , we get

$$\begin{aligned} \Delta_{2j} = & \sum_{k=0}^{2^{n-l-2}-1} sgn(n, k + j \times 2^{n-l-1}) \times \\ & (b_j b_{2^l-1-j} - b_{2^l-1+j} b_{2^l-1-1-j}) \times \\ & [(c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k})], \end{aligned} \quad (\text{B14})$$

and

$$\begin{aligned}
\Delta_{2j+1} = & - \sum_{k=2^{n-l-2}-1}^0 \text{sgn}(n, (j+1) \times 2^{n-l-1} - 1 - k) \times \\
& (b_j b_{2^l-1-j} - b_{2^{l-1}+j} b_{2^{l-1}-1-j}) \times \\
& [(c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k})].
\end{aligned} \tag{B15}$$

When l is odd, by property 2 in Appendix A, then $\Delta_{2j+1} = -\Delta_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$. Hence, $\mathcal{I}^*(a, n) = 0$. Thus, $\tau(\psi) = 0$. When l is even, by property 2 in Appendix A, then $\Delta_{2j+1} = \Delta_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$. Therefore

$$\mathcal{I}^*(a, n) = 2 \sum_{j=0}^{2^{l-2}-1} \Delta_{2j} = 2 \sum_{t=0}^{2^{l-3}-1} (\Delta_{4t} + \Delta_{4t+2}). \tag{B16}$$

By (i) of property 3 in Appendix A, from Eq. (B16),

$$\begin{aligned}
\mathcal{I}^*(a, n) = 2 & \sum_{t=0}^{2^{l-3}-1} \{[(b_{2t} b_{2^l-1-2t} - b_{2t+1} b_{2^l-2-2t}) + \\
& (b_{2^{l-1}-2-2t} b_{2^{l-1}+1+2t} - b_{2^{l-1}-1-2t} b_{2^{l-1}+2t})] \times \\
& \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}(n, k + t \times 2^{n-l}) \times \\
& [c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k}]\}.
\end{aligned} \tag{B17}$$

By (ii) of property 5 in Appendix A, from Eq. (B17), we obtain

$$\mathcal{I}^*(a, n) = 2\mathcal{I}^*(b, l)\mathcal{I}^*(c, n-l). \tag{B18}$$

Therefore, $\tau(\psi) = \tau(\phi)\tau(\omega)$.

Appendix C. The proof of theorem 2

When $l = 1$, see [11]. When $l = 2$, the proof is omitted. Next let us consider that $l \geq 3$.

Proof Step 1. Compute $\overline{\mathcal{I}}(a, n)$.

We rewrite $\overline{\mathcal{I}}(a, n)$ in Eq. (3.3) as $\overline{\mathcal{I}}(a, n) = \sum_{j=0}^{2^{l-1}-1} \Omega_j$, where

$$\begin{aligned}\Omega_j = & \sum_{i=j \times 2^{n-l-2}}^{(j+1) \times 2^{n-l-2}-1} \text{sgn}(n, i) [(a_{2i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}) \\ & - (a_{(2^{n-1}-2)-2i} a_{(2^{n-1}+1)+2i} - a_{(2^{n-1}-1)-2i} a_{2^{n-1}+2i})].\end{aligned}\quad (\text{C1})$$

By substituting the amplitudes in Eq. (B12) into Ω_{2j} and Ω_{2j+1} , we obtain

$$\begin{aligned}\Omega_{2j} = & \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}(n, k + j \times 2^{n-l-1}) \times \\ & (b_j b_{2^l-1-j} + b_{2^l-1+j} b_{2^l-1-1-j}) \times \\ & [(c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k})],\end{aligned}\quad (\text{C2})$$

and

$$\begin{aligned}\Omega_{2j+1} = & - \sum_{k=2^{n-l-2}-1}^0 \text{sgn}(n, (j+1) \times 2^{n-l-1} - 1 - k) \times \\ & (b_j b_{2^l-1-j} + b_{2^l-1+j} b_{2^l-1-1-j}) \times \\ & [(c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k})].\end{aligned}\quad (\text{C3})$$

When l is even, by property 2 in Appendix A, then $\Omega_{2j+1} = -\Omega_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$. Hence, $\overline{\mathcal{I}}(a, n) = 0$. When l is odd, by property 2 in Appendix A, then $\Omega_{2j+1} = \Omega_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$.

Therefore

$$\overline{\mathcal{I}}(a, n) = 2 \sum_{j=0}^{2^{l-2}-1} \Omega_{2j} = 2 \sum_{t=0}^{2^{l-3}-1} (\Omega_{4t} + \Omega_{4t+2}). \quad (\text{C4})$$

By (i) of property 3 in Appendix A, from Eq. (C4), we obtain

$$\begin{aligned}\overline{\mathcal{I}}(a, n) = & 2 \sum_{t=0}^{2^{l-3}-1} \{ [(b_{2t} b_{2^l-1-2t} - b_{2t+1} b_{2^l-2-2t}) - \\ & (b_{2^l-1-2-2t} b_{2^l-1+1+2t} - b_{2^l-1-1-2t} b_{2^l-1+2t})] \times \\ & \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}(n, k + t \times 2^{n-l}) \times \\ & [c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k}]\}.\end{aligned}\quad (\text{C5})$$

By (ii) of property 5 in Appendix A, from Eq. (C5), we obtain

$$\overline{\mathcal{I}}(a, n) = 2\overline{\mathcal{I}}(b, l)\mathcal{I}^*(c, n - l). \quad (\text{C6})$$

Step 2. Compute $\mathcal{I}_{+2^{n-1}}^*(a, n - 1)$.

We can rewrite $\mathcal{I}_{+2^{n-1}}^*(a, n - 1)$ as $\mathcal{I}_{+2^{n-1}}^*(a, n - 1) = \sum_{j=0}^{2^{l-1}-1} Q_j$, where

$$Q_j = \sum_{i=j \times 2^{n-l-2}}^{(j+1) \times 2^{n-l-2}-1} \text{sgn}^*(n-1, i)(a_{2^{n-1}+2i}a_{(2^n-1)-2i} - a_{2^{n-1}+1+2i}a_{(2^n-2)-2i}). \quad (\text{C7})$$

By substituting the amplitudes in Eq. (B12) into Q_{2j} and Q_{2j+1} , we get

$$Q_{2j} = \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}^*(n-1, k + j \times 2^{n-l-1}) \times (b_{2^{l-1}+j}b_{2^{l-1}-1-j})[(c_{2k}c_{(2^{n-l}-1)-2k} - c_{2k+1}c_{(2^{n-l}-2)-2k})], \quad (\text{C8})$$

and

$$Q_{2j+1} = - \sum_{k=2^{n-l-2}-1}^0 \text{sgn}^*(n-1, (j+1)2^{n-l-1} - 1 - k) \times (b_{2^{l-1}+j}b_{2^{l-1}-1-j})[(c_{2k}c_{(2^{n-l}-1)-2k} - c_{2k+1}c_{(2^{n-l}-2)-2k})]. \quad (\text{C9})$$

When l is even, by property 4 in Appendix A, then $Q_{2j+1} = -Q_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$. Hence, $\mathcal{I}_{+2^{n-1}}^*(a, n - 1) = 0$. When l is odd, by property 4 in Appendix A, then $Q_{2j+1} = Q_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$.

Therefore

$$\mathcal{I}_{+2^{n-1}}^*(a, n - 1) = 2 \sum_{j=0}^{2^{l-2}-1} Q_{2j} = 2 \sum_{t=0}^{2^{l-3}-1} (Q_{4t} + Q_{4t+2}). \quad (\text{C10})$$

By (ii) of property 3 in Appendix A, from Eq. (C10), we obtain

$$\begin{aligned}
\mathcal{I}_{+2^{n-1}}^*(a, n-1) = & \\
& 2 \sum_{t=0}^{2^{l-3}-1} [(b_{2^{l-1}+2t} b_{2^{l-1}-1-2t} - b_{2^{l-1}+1+2t} b_{2^{l-1}-2-2t}) \times \\
& \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}^*(n-1, k+t \times 2^{n-l})(c_{2k} c_{2^{n-l}-1-2k} - c_{2k+1} c_{2^{n-l}-2-2k})]. \tag{C11}
\end{aligned}$$

By (iii) of property 5 in Appendix A, from Eq. (C11), we get

$$\mathcal{I}_{+2^{n-1}}^*(a, n-1) = 2\mathcal{I}_{+2^{n-1}}^*(b, l-1)\mathcal{I}^*(c, n-l). \tag{C12}$$

Step 3. Compute $\mathcal{I}^*(a, n-1)$.

We rewrite $\mathcal{I}^*(a, n-1)$ as $\mathcal{I}^*(a, n-1) = \sum_{j=0}^{2^{l-1}-1} R_j$, where

$$R_j = \sum_{i=j \times 2^{n-l-2}}^{(j+1) \times 2^{n-l-2}-1} \text{sgn}^*(n-1, i)(a_{2i} a_{(2^{n-l}-1)-2i} - a_{2i+1} a_{(2^{n-l}-2)-2i}). \tag{C13}$$

By substituting the amplitudes in Eq. (B12) into R_{2j} and R_{2j+1} , we get

$$\begin{aligned}
R_{2j} = & \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}^*(n-1, k+j \times 2^{n-l-1}) \times \\
& (b_j b_{2^{l-1}-1-j})[(c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k})], \tag{C14}
\end{aligned}$$

and

$$\begin{aligned}
R_{2j+1} = & - \sum_{k=2^{n-l-2}-1}^0 \text{sgn}^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) \times \\
& (b_j b_{2^{l-1}-1-j})[(c_{2k} c_{(2^{n-l}-1)-2k} - c_{2k+1} c_{(2^{n-l}-2)-2k})]. \tag{C15}
\end{aligned}$$

When l is even, by property 4 in Appendix A, then $R_{2j+1} = -R_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$. Hence, $\mathcal{I}^*(a, n-1) = 0$. When l is odd, by property 4 in Appendix A, then $R_{2j+1} = R_{2j}$, $j = 0, 1, \dots, 2^{l-2} - 1$.

Therefore

$$\mathcal{I}^*(a, n-1) = 2 \sum_{j=0}^{2^{l-2}-1} R_{2j} = 2 \sum_{t=0}^{2^{l-3}-1} (R_{4t} + R_{4t+2}). \tag{C16}$$

By (ii) of property 3 in Appendix A, from Eq. (C16), we get

$$\begin{aligned} \mathcal{I}^*(a, n-1) = & \\ & 2 \sum_{t=0}^{2^{l-3}-1} [(b_{2t}b_{2^{l-1}-1-2t} - b_{2t+1}b_{2^{l-1}-2-2t}) \times \\ & \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}^*(n-1, k+t2^{n-l}) \times \\ & (c_{2k}c_{2^{n-l}-1-2k} - c_{2k+1}c_{2^{n-l}-2-2k})]. \end{aligned} \quad (\text{C17})$$

By (iii) of property 5 in Appendix A, from Eq. (C17), we get

$$\mathcal{I}^*(a, n-1) = 2\mathcal{I}^*(b, l-1)\mathcal{I}^*(c, n-l). \quad (\text{C18})$$

From steps 1, 2 and 3, it is obvious that by the definition of $\tau(\psi)$, $\tau(\psi) = 0$ whenever l is even. While l is odd, by substituting Eqs. (C6), (C12) and (C18) into $\tau(\psi)$ in Eq. (3.2),

$$\begin{aligned} \tau(\psi) &= 16|(\overline{\mathcal{I}}(b, l))^2 - 4\mathcal{I}^*(b, l-1)\mathcal{I}_{+2^{l-1}}^*(b, l-1)| \times \\ &|\mathcal{I}^*(c, n-l)|^2 = \tau(\phi)\tau^2(\omega). \end{aligned} \quad (\text{C19})$$

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